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Potential well method for Cauchy problem of generalized double dispersion equations ☆

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Abstract

In this paper we study Cauchy problem of generalized double dispersion equations $u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = f(u)_{xx}$, where $f(u) = |u|^p$, $p > 1$ or u^{2k}, $k = 1, 2, \ldots$. By introducing a family of potential wells we not only get a threshold result of global existence and nonexistence of solutions, but also obtain the invariance of some sets and vacuum isolating of solutions. In addition, the global existence and finite time blow up of solutions for problem with critical initial conditions $E(0) = d$, $I(u_0) \geqslant 0$ or $I(u_0) < 0$ are proved.

Keywords: Generalized double dispersion equations; Cauchy problem; Potential wells; Global existence; Nonexistence

1. Introduction

In the physical study of nonlinear wave propagation in waveguide, the interaction of the waveguide and the external medium and, therefore, the possibility of energy exchange through lateral surfaces of the waveguide cannot be neglected. To consider the model of interaction between the surface of a nonlinear elastic rod, whose material is hyper-elastic (e.g., the Murnaghan material), and a medium, proposed by Winkler or by Pasternak [1], the longitudinal displacement $u(x,t)$ of the rod satisfies the following double dispersion equation (DDE)

$$u_{tt} - u_{xx} = \frac{1}{4}\left(6u^2 + au_{tt} - bu_{xx}\right)_{xx} \tag{1.1}$$

which is obtained by meaning of the Hamiltonian principle (see [2,3]). Similarly, the general cubic DDE (CDDE)

$$u_{tt} - u_{xx} = \frac{1}{4}\left(cu^3 + 6u^2 + au_{tt} - bu_{xx} + du_t\right)_{xx} \tag{1.2}$$

can be obtained.

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In [4], from the Euler equation for surface wave in irrotational motion the Boussinesq equation was obtained as follows (see [5])

$$\varphi_{tt} - \varphi_{xx} - \frac{\varepsilon}{2}\varphi_{xxtt} + \frac{\varepsilon}{6}\varphi_{xxxx} - 3\epsilon\varphi_x\varphi_{xx} = 0.$$

If we let $\phi_x = w$, then the above equation becomes

$$u_{tt} - u_{xx} - \frac{\epsilon}{2}u_{xxtt} + \frac{\epsilon}{6}u_{xxxx} - \frac{3\epsilon}{2}\left(u^2\right)_{xx} = 0. \tag{1.3}$$

In [6] and [7] Chen and Wang studied the initial boundary value problem and Cauchy problem of the following generalized double dispersion equation which include above Eq. (1.2) as special cases

$$u_{tt} - u_{xx} - au_{xxtt} + bu_{xxxx} - du_{xxt} = f(u)_{xx},$$

where $a > 0$, $b > 0$ and d are constants. For the case $f'(s) \geqslant C$ (bounded below) they proved the existence of global solutions. In addition, under some conditions the global nonexistence of solutions were proved. Thus the global existence of solutions of Eq. (1.2) is resolved. However the global existence of solutions for Eqs. (1.1) and (1.3) are still open up to now.

Although there has been a lot of work using potential well method, almost all of them are on second-order nonlinear evolution equations or high-order semilinear evolution equations [8–30] and there are only a few works on higher-order strongly nonlinear evolution equations [31–34].

In this paper, we study the Cauchy problem of the generalized double dispersion equations

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} = f(u)_{xx}, \quad x \in R,\ t > 0, \tag{1.4}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in R, \tag{1.5}$$

where $f(u) = |u|^p$, $p > 1$ or u^{2k}, $k = 1, 2, \ldots$.

First for problem (1.4), (1.5) we introduce a family of potential wells W_δ and family V_δ, then by using them we not only obtain the invariant sets and vacuum isolating of solutions, but also obtain a threshold result of global existence and nonexistence of solutions. Finally the global existence and finite time blow up of solutions for problem (1.4), (1.5) with critical initial conditions $E(0) = d$, $I(u_0) \geqslant 0$ or $I(u_0) < 0$ are proved. Thus the global existence of solutions for Cauchy problem of Eqs. (1.1) and (1.3) are resolved.

In this paper: $L^p = L^p(R)$, $H^s = H^s(R)$, $\|\cdot\|_p = \|\cdot\|_{L^p(R)}$, $\|u\| = \|u\|_2$, $(u,v) = \int_{-\infty}^{\infty} uv\,\mathrm{d}x$, $(u,v)_{H^1} = (u,v) + (u_x, v_x)$.

Proposition 1.1. *(See [7].) Assume that $s > \frac{1}{2}$, $u_0 \in H^s$, $u_1 \in H^{s-1}$, $f \in C^{[s]+1}(R)$. Then problem* (1.4), (1.5) *admits a unique local solution $u \in C([0,T_m), H^s) \cap C^1([0,T_m), H^{s-1}) \cap C^2([0,T_m), H^{s-2})$, where T_m is the maximal existence time. Moreover if*

$$\limsup_{t \to T_m}\left(\left\|u(t)\right\|_{H^s} + \left\|u_t(t)\right\|_{H^{s-1}}\right) < \infty, \tag{1.6}$$

then $T_m = \infty$.

Proposition 1.2. *(See [7].) Under the assumption of Proposition* 1.1, *$T_m < \infty$ if and only if*

$$\limsup_{t \to T_m}\left\|u(t)\right\|_\infty = \infty. \tag{1.7}$$

Note that if $p > 1$, then $|s|^p \in C^{[p]}(R)$. Hence from Propositions 1.1 and 1.2 we can obtain the following

Corollary 1.3. *Let $f(u) = |u|^p$, $p > \frac{3}{2}$; $u_0 \in H^s$, $u_1 \in H^{s-1}$ for some $s > \frac{1}{2}$. Then problem* (1.4), (1.5) *admits a unique local solution $u \in C([0,T_m); H^{s_1}) \cap C^1([0,T_m); H^{s_1-1}) \cap C^2([0,T_m), H^{s_1-2})$, where $s_1 = \min\{s, p-1\}$. T_m is the maximal existence time of u. Moreover, $T_m < \infty$ if and only if* (1.7) *holds.*

Corollary 1.4. *Let $f(u) = u^{2k}, k = 1, 2, \dots,$ $u_0 \in H^s$, $u_1 \in H^{s-1}$ for some $s > \frac{1}{2}$. Then problem* (1.4), (1.5) *admits a unique local solution $u \in C([0, T_m); H^s) \cap C^1([0, T_m); H^{s-1}) \cap C^2([0, T_m), H^{s-2})$, where T_m is the maximal existence time of u. Moreover, $T_m < \infty$ if and only if* (1.7) *holds.*

2. Preliminary lemmas and introducing of families $\{W_\delta\}$ and $\{V_\delta\}$

In this section we first give some preliminary lemmas, then by using them we introduce two families $\{W_\delta\}$ and $\{V_\delta\}$. Firstly for problem (1.4), (1.5) we define

$$J(u) = \frac{1}{2}\|u\|_{H^1}^2 + \int_{-\infty}^{\infty} F(u)\,dx,$$

$$I(u) = \|u\|_{H^1}^2 + \int_{-\infty}^{\infty} uf(u)\,dx,$$

$$I_\delta(u) = \delta\|u\|_{H^1}^2 + \int_{-\infty}^{\infty} uf(u)\,dx, \quad \delta > 0.$$

Lemma 2.1. *Let $f(u) = |u|^p$, $p > 1$. Then*

(i) $uf(u) > 0$, $F(u) > 0$ *for* $u > 0$; $uf(u) < 0$, $F(u) < 0$ *for* $u < 0$.
(ii) $f(\lambda u) = \lambda^p f(u)$, $F(\lambda u) = \lambda^{p+1} F(u)$, $\forall u \in \mathbb{R}$, $\lambda > 0$.
(iii) $|uf(u)| = |u|^{p+1}$, $|F(u)| = \frac{1}{p+1}|u|^{p+1}$, $\forall u \in \mathbb{R}$.
(iv) $(p+1)F(u) = uf(u)$, $\forall u \in \mathbb{R}$, *where*

$$F(u) = \int_0^u f(s)\,ds.$$

Definition 2.2. We define

$$H^- = \{u \in H^1 \mid u(x) \leqslant 0,\ \|u\|_{H^1} \neq 0\},$$
$$\Omega^+(u) = \{x \in \mathbb{R} \mid u(x) > 0\},$$
$$\Omega^-(u) = \{x \in \mathbb{R} \mid u(x) < 0\},$$
$$\Omega^0(u) = \{x \in \mathbb{R} \mid u(x) = 0\}.$$

Lemma 2.3. *Let $f(u) = |u|^p$, $p > 1$, $u \in H^1$, and*

$$\varphi(\lambda) = -\frac{1}{\lambda}\int_{-\infty}^{\infty} uf(\lambda u)\,dx.$$

Assume that $\int_{-\infty}^{\infty} uf(u)\,dx < 0$. Then

(i) $\varphi(\lambda)$ *is increasing on* $0 < \lambda < \infty$.
(ii) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$, $\lim_{\lambda\to+\infty}\varphi(\lambda) = +\infty$.

Proof. This lemma follows from

$$\varphi(\lambda) = -\frac{1}{\lambda}\int_{-\infty}^{\infty} uf(\lambda u)\,dx = -\lambda^{p-1}\int_{-\infty}^{\infty} uf(u)\,dx. \qquad \square$$

Lemma 2.4. *Let $f(u) = |u|^p$, $p > 1$, $u \in H^1$. Then*

(i) $\lim_{\lambda \to 0} J(\lambda u) = 0$.
(ii) $I(\lambda u) = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u)$, $\forall \lambda > 0$.

Furthermore if $\int_{-\infty}^{\infty} u f(u)\,\mathrm{d}x < 0$, then

(iii) $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$.
(iv) *In the interval $0 < \lambda < \infty$ there exists a unique $\lambda^* = \lambda^*(u)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u)\Big|_{\lambda=\lambda^*} = 0.$$

(v) *$J(\lambda u)$ is increasing on $0 \leqslant \lambda \leqslant \lambda^*$, decreasing on $\lambda^* \leqslant \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$.*
(vi) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$ and $I(\lambda^* u) = 0$.*

Proof. Parts (i)–(iii) are obvious.

Note that $\int_{-\infty}^{\infty} u f(u)\,\mathrm{d}x \neq 0$ implies $\|u\|_{H^1} \neq 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u) = \lambda\big(\|u\|_{H_1}^2 - \varphi(\lambda)\big), \tag{2.1}$$

which gives parts (iv) and (v).

Part (vi) follows from part (ii) and (2.1). □

Corollary 2.5. *If in Lemmas 2.3 and 2.4 the assumption $\int_{-\infty}^{\infty} u f(u)\,\mathrm{d}x < 0$ is replaced by $u \in H^-$, then the conclusions of Lemmas 2.3 and 2.4 also hold.*

Lemma 2.6. *Let $f(u) = |u|^p$, $p > 1$. Assume that $u \in H^1$ and $0 < \|u\|_{H^1} < r(\delta)$. Then $I_\delta(u) > 0$. In particular, if $0 < \|u\|_{H_1} < r(1)$, then $I(u) > 0$, where*

$$r(\delta) = \left(\frac{\delta}{C_*^{p+1}}\right)^{\frac{1}{p-1}},$$

C_* is imbedding constant from H^1 into L^{p+1}.

Proof. From $0 < \|u\|_{H^1} < r(\delta)$ we get

$$\begin{aligned} -\int_{-\infty}^{\infty} u f(u)\,\mathrm{d}x &\leqslant \int_{-\infty}^{\infty} \big|u f(u)\big|\,\mathrm{d}x = \int_{-\infty}^{\infty} |u|^{p+1}\,\mathrm{d}x = \|u\|_{p+1}^{p+1} \\ &\leqslant C_*^{p+1}\|u\|_{H^1}^{p+1} = C_*^{p+1}\|u\|_{H^1}^{p-1}\|u\|_{H^1}^2 < \delta\|u\|_{H^1}^2, \end{aligned}$$

which gives $I_\delta(u) > 0$. □

Lemma 2.7. *Let $f(u) = |u|^p$, $p > 1$. Assume that $u \in H^1$ and $I_\delta(u) < 0$. Then $\|u\|_{H^1} > r(\delta)$. In particular, if $I(u) < 0$, then $\|u\|_{H^1} > r(1)$.*

Proof. First $I_\delta(u) < 0$ implies $\|u\|_{H^1} \neq 0$. From this and

$$\delta\|u\|_{H^1}^2 < -\int_{-\infty}^{\infty} u f(u)\,\mathrm{d}x \leqslant C_*^{p+1}\|u\|_{H^1}^{p-1}\|u\|_{H^1}^2$$

we get $\|u\|_{H^1} > r(\delta)$. □

Lemma 2.8. *Let $f(u)=|u|^p$, $p>1$. Assume that $u\in H^1$, $I_\delta(u)=0$ and $\|u\|_{H^1}\neq 0$. Then $\|u\|_{H^1}\geqslant r(\delta)$. In particular, if $I(u)=0$ and $\|u\|_{H^1}\neq 0$, then $\|u\|_{H^1}\geqslant r(1)$.*

Proof. From $I_\delta(u)=0$ we have

$$\delta\|u\|_{H^1}^2=-\int_{-\infty}^{\infty}uf(u)\,\mathrm{d}x\leqslant C_*^{p+1}\|u\|_{H^1}^{p-1}\|u\|_{H^1}^2,$$

which together with $\|u\|_{H^1}\neq 0$ gives $\|u\|_{H^1}\geqslant r(\delta)$. $\square$

Lemma 2.9. *Let $f(u)=|u|^p$, $p>1$. Assume that $u\in H^1$, $I_\delta(u)=0$ and $\|u\|_{H^1}\neq 0$. Then*

(i) $J(u)>0$ *for* $0<\delta<\frac{p+1}{2}$;
(ii) $J(u)=0$ *for* $\delta=\frac{p+1}{2}$;
(iii) $J(u)<0$ *for* $\delta>\frac{p+1}{2}$.

Proof. This lemma follows from Lemma 2.8 and

$$\begin{aligned}J(u)&=\frac{1}{2}\|u\|_{H^1}^2+\int_{-\infty}^{\infty}F(u)\,\mathrm{d}x=\frac{1}{2}\|u\|_{H^1}^2+\frac{1}{p+1}\int_{-\infty}^{\infty}uf(u)\,\mathrm{d}x\\&=\left(\frac{1}{2}-\frac{\delta}{p+1}\right)\|u\|_{H^1}^2+\frac{1}{p+1}I_\delta(u)=\left(\frac{1}{2}-\frac{\delta}{p+1}\right)\|u\|_{H^1}^2.\qquad\square\end{aligned}$$

Definition 2.10. For $f(u)=|u|^p$, $p>1$, we define

(i)

$$d=\inf_{u\in H^1,\,u\neq 0}\Bigl(\sup_{\lambda>0}J(\lambda u)\Bigr)\tag{2.2}$$

or equivalently

$$d=\inf_{u\in\mathcal{N}}J(u),\quad \mathcal{N}=\bigl\{u\in H^1\bigm| I(u)=0,\ \|u\|_{H^1}\neq 0\bigr\}.$$

(ii)

$$d(\delta)=\inf_{u\in\mathcal{N}_\delta}J(u),\quad \delta>0,\ \mathcal{N}_\delta=\bigl\{u\in H^1\bigm| I_\delta(u)=0,\ \|u\|_{H^1}\neq 0\bigr\}.$$

(iii)

$$d_1=\inf_{u\in\mathcal{N}^-}J(u),\quad \mathcal{N}^-=\bigl\{u\in H^-\bigm| I(u)=0\bigr\}.$$

Lemma 2.11. *Let $f(u)=|u|^p$, $p>1$. Assume that $u\in H^1$ and $I(u)<0$. Then*

$$d<\frac{p-1}{2(p+1)}\|u\|_{H^1}^2.\tag{2.3}$$

Proof. First $I(u)<0$ gives $-\int_{-\infty}^{\infty}uf(u)\,\mathrm{d}x>\|u\|_{H^1}^2>0$. Hence from Lemma 2.4 it follows that $J(\lambda u)$ takes the maximum at $\lambda^*=\lambda^*(u)$, which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)\Big|_{\lambda=\lambda^*}=0.\tag{2.4}$$

Let

$$a(u) = \|u\|_{H^1}^2, \qquad b(u) = -\int_{-\infty}^{\infty} uf(u)\,\mathrm{d}x.$$

Then

$$J(\lambda u) = \frac{\lambda^2}{2}a(u) - \frac{\lambda^{p+1}}{p+1}b(u).$$

Note (2.4) gives

$$\lambda^* = \left(\frac{a(u)}{b(u)}\right)^{\frac{1}{p-1}}.$$

Therefore from Definition 2.10 we get

$$\begin{aligned} d &\leqslant \sup_{\lambda>0} J(\lambda u) = J(\lambda^* u) \\ &= \frac{1}{2}\left(\frac{a(u)}{b(u)}\right)^{\frac{2}{p-1}} a(u) - \frac{1}{p+1}\left(\frac{a(u)}{b(u)}\right)^{\frac{p+1}{p-1}} b(u) \\ &= \frac{p-1}{2(p+1)}\left(\frac{a(u)}{b(u)}\right)^{\frac{2}{p-1}} a(u) < \frac{p-1}{2(p+1)}a(u) = \frac{p-1}{2(p+1)}\|u\|_{H^1}^2. \qquad \square \end{aligned}$$

Lemma 2.12. *Let $f(u) = |u|^p$, $p > 1$, $w \in \mathcal{N}$ be a minimizer, i.e.,*

$$J(w) = d.$$

Then $w \in \mathcal{N}^-$.

Proof. In order to prove $w \in \mathcal{N}^-$ it is enough to prove $|\Omega^+(w)| = 0$. If it is false, then $|\Omega^+(w)| > 0$. We want to show it is impossible by considering the following two cases:

(i) $|\Omega^+(w)| > 0$, $|\Omega^-(w)| = 0$.
In this case we have $\int_{-\infty}^{\infty} wf(w)\,\mathrm{d}x = \int_{\Omega^+(w)\cup\Omega^-(w)\cup\Omega^0(w)} wf(w)\,\mathrm{d}x = \int_{\Omega^+(w)} wf(w)\,\mathrm{d}x > 0$ and $I(w) > 0$, which contradicts $w \in \mathcal{N}$.

(ii) $|\Omega^+(w)| > 0$, $|\Omega^-(w)| > 0$.
In this case we have $-|w| \in H^-$ and $I(-|w|) < I(w) = 0$. From Corollary 2.5 it follows that there exists a unique λ_1 such that $0 < \lambda_1 < 1$ and $-\lambda_1|w| \in \mathcal{N}^-$. On the other hand, from (2.2) we get

$$d = J(w) = \sup_{\lambda>0} J(\lambda w). \tag{2.5}$$

Hence we have

$$d = J(w) \geqslant J(\lambda_1 w) > J(-\lambda_1|w|) \geqslant d_1,$$

which contradicts the definition of d and d_1. $\square$

Theorem 2.13. *Let $f(u) = |u|^p$, $p > 1$. Then*

(i) *$d(\delta) \geqslant a(\delta)r^2(\delta)$ for $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$, $0 < \delta < \frac{p+1}{2}$. In particular, we have $\frac{1}{\alpha C_*^\alpha} = a(1)r^2(1) \leqslant d(1) = d \leqslant d_1 < \infty$, $\alpha = \frac{2(p+1)}{p-1}$.*

(ii) *$d(\delta) = (\frac{1}{2} - \frac{\delta}{p+1})\delta^{\frac{2}{p-1}}\frac{2(p+1)}{p-1}d$.*

Proof. (i) For any $u \in \mathcal{N}_\delta$ we have $\|u\|_{H^1} \geqslant r(\delta)$ and

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_{H^1}^2 + \int_{-\infty}^{\infty} F(u)\,\mathrm{d}x = \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{p+1}\int_{-\infty}^{\infty} uf(u)\,\mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|u\|_{H^1}^2 + \frac{1}{p+1}I_\delta(u) = a(\delta)\|u\|_{H^1}^2 \\ &\geqslant a(\delta)r^2(\delta), \quad 0 < \delta < \frac{p+1}{2}, \end{aligned}$$

which gives $d(\delta) \geqslant a(\delta)r^2(\delta)$ for $0 < \delta < \frac{p+1}{2}$.

(ii) (a) Let $u \in \mathcal{N}$ be a minimizer, i.e.,

$$J(u) = d.$$

For any $\delta > 0$ define $\lambda = \lambda(\delta)$ by

$$\delta\|\lambda u\|_{H^1}^2 = -\int_{-\infty}^{\infty} \lambda u f(\lambda u)\,\mathrm{d}x, \tag{2.6}$$

i.e.,

$$\delta\|u\|_{H^1}^2 = \lambda^{p-1}\left(-\int_{-\infty}^{\infty} uf(u)\,\mathrm{d}x\right). \tag{2.7}$$

Then for each $\delta > 0$ there exists a unique

$$\lambda(\delta) = \left(\frac{\delta a(u)}{b(u)}\right)^{\frac{1}{p-1}}$$

satisfying (2.7) and (2.6) which implies $\lambda(\delta)u \in \mathcal{N}_\delta$, where $a(u)$ and $b(u)$ are defined in Lemma 2.11. Since $u \in \mathcal{N}$ implies $a(u) = b(u)$ we have

$$\lambda(\delta) = \delta^{\frac{1}{p-1}}.$$

Therefore

$$\begin{aligned} d(\delta) &\leqslant J\big(\lambda(\delta)u\big) = \frac{1}{2}\delta^{\frac{2}{p-1}}a(u) - \frac{1}{p+1}\delta^{\frac{p+1}{p-1}}b(u) \\ &= \delta^{\frac{2}{p-1}}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)a(u). \end{aligned}$$

Note that

$$J(u) = \frac{1}{2}a(u) - \frac{1}{p+1}b(u) = \frac{p-1}{2(p+1)}a(u),$$

we get

$$\begin{aligned} d(\delta) &\leqslant \delta^{\frac{2}{p-1}}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1}J(u) \\ &= \delta^{\frac{2}{p-1}}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1}d \end{aligned}$$

(b) Let $\delta > 0$, $u \in \mathcal{N}_\delta$ be a minimizer, i.e.,

$$J(u) = d(\delta).$$

Define $\lambda = \lambda(\delta)$ by

$$\|\lambda u\|_{H^1}^2 = -\int_{-\infty}^{\infty} \lambda u f(\lambda u)\,\mathrm{d}x, \tag{2.8}$$

i.e.,

$$\|u\|_{H^1}^2 = \lambda^{p-1}\left(-\int_{-\infty}^{\infty} u f(u)\,\mathrm{d}x\right),$$

$$\lambda = \left(\frac{a(u)}{b(u)}\right)^{\frac{1}{p-1}}.$$

Since $u \in \mathcal{N}_\delta$ implies $\delta a(u) = b(u)$ we have

$$\lambda = \left(\frac{1}{\delta}\right)^{\frac{1}{p-1}}.$$

From $\lambda(\delta)u \in \mathcal{N}$ we get

$$\begin{aligned} d \leqslant J\big(\lambda(\delta)u\big) &= \frac{1}{2}\left(\frac{1}{\delta}\right)^{\frac{2}{p-1}} a(u) - \frac{1}{p+1}\left(\frac{1}{\delta}\right)^{\frac{p+1}{p-1}} b(u) \\ &= \left(\frac{1}{\delta}\right)^{\frac{2}{p-1}}\left(\frac{1}{2}a(u) - \frac{1}{p+1}\left(\frac{1}{\delta}\right)\delta a(u)\right) = \left(\frac{1}{\delta}\right)^{\frac{2}{p-1}} \frac{p-1}{2(p+1)} a(u). \end{aligned}$$

Note that

$$J(u) = \frac{1}{2}a(u) - \frac{1}{p+1}b(u) = \left(\frac{1}{2} - \frac{\delta}{p+1}\right)a(u),$$

we get

$$\begin{aligned} d &\leqslant \left(\frac{1}{\delta}\right)^{\frac{2}{p-1}} \frac{p-1}{p+1-2\delta} J(u) \\ &= \left(\frac{1}{\delta}\right)^{\frac{2}{p-1}} \frac{p-1}{p+1-2\delta} d(\delta), \end{aligned}$$

which gives

$$d(\delta) \geqslant \delta^{\frac{2}{p-1}}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1} d.$$

From (a) and (b) we obtain (ii). □

Corollary 2.14. *Let $f(u) = |u|^p$, $p > 1$. Then*

(i) $\lim_{\delta \to 0} d(\delta) = 0$, $d(\frac{p+1}{2}) = 0$;
(ii) *$d(\delta)$ is continuous on $0 < \delta \leqslant \frac{p+1}{2}$;*
(iii) *$d(\delta)$ is increasing on $0 < \delta \leqslant 1$, decreasing on $1 \leqslant \delta \leqslant \frac{p+1}{2}$ and takes the maximum $d = d(1)$ at $\delta = 1$.*

Proof. (i) and (ii) follow immediately from (ii) of Theorem 2.13.
(iii) follows from

$$d'(\delta) = A\big((p+1-2\delta)\delta^{\frac{2}{p-1}}\big)' = \frac{2(p+1)}{p-1} A \delta^{\frac{3-p}{p-1}}(1-\delta),$$

where

$$A = \frac{d}{p-1}. \qquad \square$$

Definition 2.15. Now we can define

$$W = \{u \in H^1 \mid I(u) > 0,\ J(u) < d\} \cup \{0\};$$
$$V = \{u \in H^1 \mid I(u) < 0,\ J(u) < d\};$$
$$W_\delta = \{u \in H^1 \mid I_\delta(u) > 0,\ J(u) < d(\delta)\} \cup \{0\}, \quad 0 < \delta < \frac{p+1}{2};$$
$$V_\delta = \{u \in H^1 \mid I_\delta(u) < 0,\ J(u) < d(\delta)\}, \quad 0 < \delta < \frac{p+1}{2}.$$

3. Invariant sets and vacuum isolating of solutions

In this section we discuss the invariance of some sets under the flow of (1.4), (1.5) and vacuum isolating of solutions for problem (1.4), (1.5). First we give the energy equality for problem (1.4), (1.5). From Lemma 3.1 in [7] we can obtain the following

Lemma 3.1. *Let $f(u) = |u|^p$, $p \geqslant 2$, $u_0 \in H^1$, $u_1 \in L^2$, $\wedge^{-1}u_1 \in L^2$, $u \in C([0, T_m); H^1) \cap C^1([0, T_m); L^2) \cap C^2([0, T_m), H^{-1})$ be the weak solution of problem* (1.4), (1.5)*, where T_m is the maximal existence time of u. Then we have*

$$E(t) \equiv E(0) \quad \textit{for all } t \in [0, T_m), \tag{3.1}$$

where

$$E(t) = \frac{1}{2}\left\|\wedge^{-1}u_t\right\|^2 + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_{H^1}^2 + \int_{-\infty}^{\infty} F(u)\,\mathrm{d}x \equiv \frac{1}{2}\left\|\wedge^{-1}u_t\right\|^2 + \frac{1}{2}\|u_t\|^2 + J(u),$$

$\wedge^{-\alpha}\varphi = \mathscr{F}^{-1}[|\xi|^{-\alpha}\mathscr{F}\varphi(\xi)]$, $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote Fourier transformation and inverse transformation on $\mathbb{R}$, respectively.

In the following we first consider the case $0 < E(0) < d$.

Theorem 3.2. *Let $f(u) = |u|^p$, $p \geqslant 2$, $u_0 \in H^1$, $u_1 \in L^2$ and $\wedge^{-1}u_1 \in L^2$. Assume that $0 < e < d$, (δ_1, δ_2) is the maximal interval such that $d(\delta) > e$ for $\delta \in (\delta_1, \delta_2)$. Then:*

(i) *All weak solutions of problem* (1.4), (1.5) *with $E(0) = e$ belong to W_δ for $\delta \in (\delta_1, \delta_2)$, provided $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$.*
(ii) *All weak solutions of problem* (1.4), (1.5) *with $E(0) = e$ belong to V_δ for $\delta \in (\delta_1, \delta_2)$, provided $I(u_0) < 0$.*

Proof. (i) Let $u(t)$ be any weak solution of problem (1.4), (1.5) with $E(0) = e$, $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$, T_m be the maximal existence time of $u(t)$. First we prove $u_0 \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$. In fact, if $\|u_0\|_{H^1} = 0$, then $u_0 \in W_\delta$ for $\delta \in (0, \frac{p+1}{2})$. If $I(u_0) > 0$, then $\|u_0\|_{H^1} \neq 0$. From

$$\frac{1}{2}\left\|\wedge^{-1}u_1\right\|^2 + \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) = e < d(\delta), \quad \delta \in (\delta_1, \delta_2), \tag{3.2}$$

we can get $J(u_0) < d(\delta)$ and $I_\delta(u_0) > 0$ for $\delta \in (\delta_1, \delta_2)$. Otherwise there exists $\bar{\delta} \in (\delta_1, \delta_2)$ such that $I_{\bar{\delta}}(u_0) = 0$ and $J(u_0) \geqslant d(\bar{\delta})$ which contradicts (3.2). Hence we have $u_0 \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$. Next we prove $u(t) \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t \in (0, T_m)$. If it is false, then there exists $t_0 \in (0, T_m)$ such that $u(t_0) \in \partial W_\delta$ for some $\delta \in (\delta_1, \delta_2)$, i.e., $I_\delta(u(t_0)) = 0$, $\|u(t_0)\|_{H^1} \neq 0$ or $J(u(t_0)) = d(\delta)$. From (3.1) we get

$$\frac{1}{2}\|\wedge^{-1}u_t\|^2 + \frac{1}{2}\|u_t\|^2 + J(u) \equiv E(0) = e < d(\delta), \quad \delta \in (\delta_1, \delta_2),\ t \in (0, T_m). \tag{3.3}$$

Hence $J(u(t_0)) = d(\delta)$ is impossible. If $I_\delta(u(t_0)) = 0$, $\|u(t_0)\|_{H^1} \neq 0$, then we have $J(u(t_0)) \geqslant d(\delta)$ which contradicts (3.3).

(ii) Let $u(t)$ be any weak solution of problem (1.4), (1.5) with $E(0) = e$, $I(u_0) < 0$, T_m be the maximal existence time of $u(t)$. First from $I(u_0) < 0$ and (3.2) we can prove $u_0 \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$. Next we prove $u(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t \in (0, T_m)$. If it is false, then there exits $t_0 \in (0, T_m)$ such that $u(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t \in [0, t_0)$ and $u(t_0) \in \partial V_\delta$ for some $\delta \in (\delta_1, \delta_2)$, i.e., $I_\delta(u(t_0)) = 0$ or $J(u(t_0)) = d(\delta)$. Again (3.3) shows that $J(u(t_0)) = d(\delta)$ is impossible. If $I_\delta(u(t_0)) = 0$, then form $I_\delta(u) < 0$ for $0 < t < t_0$ we get $\|u\|_{H^1} > r(\delta)$ for $0 \leqslant t < t_0$ and $\|u(t_0)\|_{H^1} \geqslant r(\delta)$. Hence we have $J(u(t_0)) \geqslant d(\delta)$ which also contradicts (3.3). □

Corollary 3.3. *If in Theorem* 3.2 *the assumption* $E(0) = e$ *is replaced by* $0 < E(0) \leqslant e$, *then the conclusion of Theorem* 3.2 *also holds.*

Theorem 3.4. *Let* $f(u)$, $u_i(x)$ $(i = 0, 1)$, *e and* (δ_1, δ_2) *be the same as those in Theorem* 3.2. *Then for any* $\delta \in (\delta_1, \delta_2)$ *both sets* W_δ *and* V_δ *are invariant, thereby both sets*

$$W_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} W_\delta \quad \text{and} \quad V_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} V_\delta$$

are invariant under the flow of (1.4), (1.5), *respectively, provide* $0 < E(0) \leqslant e$.

From (3.3) we see that if $0 < E(0) \leqslant e < d$, (δ_1, δ_2) is the maximal interval such that $d(\delta) > e$ for $\delta \in (\delta_1, \delta_2)$, then for any $\delta \in (\delta_1, \delta_2)$, $I_\delta(u) = 0$ and $\|u\|_{H^1} \neq 0$ is impossible. Therefore for the set of all solutions of problem (1.4), (1.5) with $0 < E(0) \leqslant e < d$, there exists a vacuum region

$$U_e = \mathcal{N}_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} \mathcal{N}_\delta = \left\{u \in H^1 \;\middle|\; \|u\|_{H^1} \neq 0,\ I_\delta(u) = 0,\ \delta_1 < \delta < \delta_2\right\}$$

such that $u \notin U_e$ for all solutions u of problem (1.4), (1.5) with $0 < E(0) \leqslant e < d$.

Next we consider the case $E(0) \leqslant 0$.

Lemma 3.5. *Let* $f(u) = |u|^p$, $p \geqslant 2$, $u_0 \in H^1$, $u_1 \in L^2$ *and* $\wedge^{-1}u_1 \in L^2$. *Then all nontrivial weak solutions of problem* (1.4), (1.5) *with* $E(0) = 0$ *satisfy*

$$\|u\|_{H^1} \geqslant r_0 = \left(\frac{p+1}{2C_*^{p+1}}\right)^{\frac{1}{p-1}}.$$

Proof. Let $u(t)$ be any weak solution of problem (1.4), (1.5) with $E(0) = 0$, T_m be the maximal existence time of $u(t)$. First from

$$\frac{1}{2}\|\wedge^{-1}u_t\|^2 + \frac{1}{2}\|u_t\|^2 + J(u) \equiv E(0) = 0,$$

we get $J(u) \leqslant 0$ for $0 \leqslant t < T_m$. Hence we have

$$\begin{aligned}
\frac{1}{2}\|u\|_{H^1}^2 &\leqslant -\int_{-\infty}^{\infty} F(u)\,dx \leqslant \int_{-\infty}^{\infty} \left|F(u)\right|dx = \frac{1}{p+1}\int_{-\infty}^{\infty} |u|^{p+1}\,dx \\
&= \frac{1}{p+1}\|u\|_{p+1}^{p+1} \leqslant \frac{1}{p+1}C_*^{p+1}\|u\|_{H^1}^{p+1} \\
&= \frac{1}{p+1}C_*^{p+1}\|u\|_{H^1}^{p-1}\|u\|_{H^1}^2, \quad 0 \leqslant t < T_m,
\end{aligned}$$

which gives that either $\|u\|_{H^1} \geqslant r_0$ or $\|u\|_{H^1} = 0$ for $0 \leqslant t < T_m$. If $\|u_0\|_{H^1} = 0$, then $\|u\|_{H^1} \equiv 0$ for $0 \leqslant t < T_m$. Otherwise there exists $t_0 \in (0, T_m)$ such that $0 < \|u(t_0)\|_{H^1} < r_0$ which contradicts above conclusion. By a similar argument, we can prove that if $\|u_0\|_{H^1} \neq 0$, then $\|u\|_{H^1} \geqslant r_0$ for $0 \leqslant t < T_m$. □

Theorem 3.6. *Let $f(u) = |u|^p$, $p \geqslant 2$, $u_0 \in H^1$, $u_1 \in L^2$ and $\wedge^{-1}u_1 \in L^2$. Assume that $E(0) < 0$ or $E(0) = 0$, $\|u_0\|_{H^1} \neq 0$. Then all weak solutions of problem* (1.4), (1.5) *belong to V_δ for $\delta \in (0, \frac{p+1}{2})$.*

Proof. Let $u(t)$ be any weak solution of problem (1.4), (1.5) with $E(0) < 0$ or $E(0) = 0$, $\|u_0\|_{H^1} \neq 0$, T_m be the maximal existence time of $u(t)$. From (3.1) we get

$$\begin{aligned} &\frac{1}{2}\left\|\wedge^{-1}u_t\right\|^2 + \frac{1}{2}\|u_t\|^2 + \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|u\|_{H^1}^2 + \frac{1}{p+1}I_\delta(u) \\ &\quad = \frac{1}{2}\left\|\wedge^{-1}u_t\right\|^2 + \frac{1}{2}\|u_t\|^2 + J(u) \equiv E(0), \quad 0 < \delta < \frac{p+1}{2},\ 0 \leqslant t < T_m. \end{aligned} \tag{3.4}$$

From (3.4) it follows that if $E(0) < 0$, then $I_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $\delta \in (0, \frac{p+1}{2})$, $0 \leqslant t < T_m$. If $E(0) = 0$ and $\|u_0\|_{H^1} \neq 0$, then by Lemma 3.5 we have $\|u\|_{H^1} \geqslant r_0 > 0$ for $0 \leqslant t < T_m$. Hence form (3.4) we again obtain $I_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $\delta \in (0, \frac{p+1}{2})$, $t \in [0, T_m)$. Therefore for above two cases we always have $u(t) \in V_\delta$ for $\delta \in (0, \frac{p+1}{2})$, $t \in [0, T_m)$. □

4. Global existence and finite time blow up of solutions

In this section we prove the global existence and finite time blow up of solutions and give a threshold result of global existence and nonexistence of solutions for problem (1.4), (1.5) with $E(0) < d$.

Theorem 4.1. *Let $f(u) = |u|^p$, $p \geqslant 2$, $u_0 \in H^s$, $u_1 \in H^{s-1}$ for some $s \geqslant 1$ and $\wedge^{-1}u_1 \in L^2$. Assume that $E(0) < d$, $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$. Then problem* (1.4), (1.5) *admits a unique global solution $u \in C([0,\infty); H^{s_1}) \cap C^1([0,\infty), H^{s_1-1}) \cap C^2([0,\infty), H^{s_1-2})$ and $u \in W$ for $0 \leqslant t < \infty$, where $s_1 = \min\{s, p-1\}$.*

Proof. First form Corollary 1.3 it follows that problem (1.4), (1.5) admits a unique local solution $u \in C([0,T_m), H^{s_1}) \cap C^1([0,T_m), H^{s_1-1}) \cap C^2([0,T_m), H^{s_1-2})$ satisfying

$$\frac{1}{2}\left\|\wedge^{-1}u_t\right\|^2 + \frac{1}{2}\|u_t\|^2 + J(u) \equiv E(0), \quad 0 \leqslant t < T_m, \tag{4.1}$$

where T_m is the maximal existence time of u. Equation (4.1) gives

$$\frac{1}{2}\left\|\wedge^{-1}u_t\right\|^2 + \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 + \frac{1}{p+1}I(u) \equiv E(0), \quad 0 \leqslant t < T_m. \tag{4.2}$$

Note that from

$$\frac{1}{2}\left\|\wedge^{-1}u_1\right\|^2 + \frac{1}{2}\|u_1\|^2 + \frac{p-1}{2(p+1)}\|u_0\|_{H^1}^2 + \frac{1}{p+1}I(u_0) = E(0),$$

it follows that if $I(u_0) > 0$, then $E(0) > 0$; if $\|u_0\|_{H^1} = 0$, then $E(0) = \frac{1}{2}\| \wedge^{-1} u_1\|^2 + \frac{1}{2}\|u_1\|^2 \geqslant 0$. If $E(0) > 0$, then by Theorem 3.2 we have $u \in W$ for $0 \leqslant t < T_m$. Hence from (4.2) we get

$$\left\|\wedge^{-1}u_t\right\|^2 + \|u_t\|^2 + \|u\|_{H^1}^2 \leqslant \frac{2(p+1)}{p-1}E(0), \quad 0 \leqslant t < T_m,$$

and $\|u\|_\infty \leqslant C$, $0 \leqslant t < T_m$. Hence $T_m = +\infty$. If $\|u_0\|_{H^1} = 0$, $E(0) = 0$, then by Lemma 3.5 we have $\|u\|_{H^1} \equiv 0$ for $0 \leqslant t < T_m$ and $T_m = +\infty$. □

Corollary 4.2. *Let $f(u) = u^{2k}$, $k = 1, 2, \ldots$, $u_0 \in H^s$, $u_1 \in H^{s-1}$ for some $s \geqslant 1$ and $\wedge^{-1}u_1 \in L^2$. Assume that $E(0) < d$, $I(u_0) > 0$ or $\|u_0\|_{H^1} = 0$. Then problem* (1.4), (1.5) *admits a unique global solution $u \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1}) \cap C^2([0,\infty); H^{s-2})$ and $u \in W$ for $0 \leqslant t < \infty$.*

Proof. First from Corollary 1.4 it follows that problem (1.4), (1.5) admits a unique local solution $u \in C([0,T_m); H^s) \cap C^1([0,T_m); H^{s-1}) \cap C^2([0,T_m); H^{s-2})$ satisfying (4.1), where T_m is the maximal existence time of u. Equation (4.1) gives (4.2) and $T_m = +\infty$. □

Corollary 4.3. *If in Theorem* 4.1 *and Corollary* 4.2 *the assumption "$E(0) < d$, $I(u_0) > 0$" is replaced by "$0 < E(0) < d$, $I_{\delta_2}(u_0) > 0$," where (δ_1, δ_2) is the maximal interval such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$. Then for the global solution in Theorem* 4.1 *and Corollary* 4.2 *we further have $u \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $0 \leqslant t < \infty$.*

Theorem 4.4. *If in Corollary* 4.3 *the assumption "$I_{\delta_2}(u_0) > 0$ or $\|u_0\|_{H^1} = 0$" is replaced by "$\|u_0\|_{H^1} < r(\delta_2)$," then problem* (1.4), (1.5) *admits a global solution given in Corollary* 4.3 *satisfying*

$$\|u\|_{H^1}^2 \leqslant \frac{E(0)}{a(\delta_1)}, \quad \|\wedge^{-1}u_t\|^2 + \|u_t\|^2 \leqslant 2E(0), \quad 0 \leqslant t < \infty. \tag{4.3}$$

Proof. First $\|u_0\|_{H^1} < r(\delta_2)$ gives $I_{\delta_2}(u_0) > 0$ or $\|u_0\|_{H^1} = 0$. Hence from Corollary 4.3 it follows that problem (1.4), (1.5) admits a unique global solution u given in Corollary 4.3 satisfying $u \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $0 \leqslant t < \infty$. In

$$\frac{1}{2}\|\wedge^{-1}u_t\|^2 + \frac{1}{2}\|u_t\|^2 + a(\delta)\|u\|_{H^1}^2 + \frac{1}{p+1}I_\delta(u) = \frac{1}{2}\|\wedge^{-1}u_t\|^2 + \frac{1}{2}\|u_t\|^2 + J(u) \equiv E(0),$$

letting $\delta \to \delta_1$ we get (4.3). □

In the following we consider the global nonexistence and finite time blow up of solution for problem (1.4), (1.5).

Theorem 4.5. *Let $f(u) = |u|^p$, $p \geqslant 2$, $u_0 \in H^1$, $u_1 \in L^2$ and $\wedge^{-1}u_1 \in L^2$. Assume that $E(0) < d$ and $I(u_0) < 0$. Then the existence time of weak solution for problem* (1.4), (1.5) *is finite.*

Proof. First Corollary 1.3 gives the existence of local weak solution $u \in C([0, T_m); H^1) \cap C^1([0, T_m); L^2) \cap C^2([0, T_m), H^{-1})$ with $\wedge^{-1}u_t \in C([0, T_m); L^2)$ satisfying

$$\wedge^{-2}u_{tt} + u_{tt} + u - u_{xx} + f(u) = 0, \quad 0 < t < T_m, \tag{4.4}$$

where T_m is the maximal existence time of u. We prove $T_m < \infty$. If it is false, then $T_m = +\infty$. Let

$$\phi(t) = \|\wedge^{-1}u\|^2 + \|u\|^2.$$

Then

$$\begin{aligned}
\dot\phi(t) &= 2(\wedge^{-1}u_t, \wedge^{-1}u) + 2(u_t, u), \\
\ddot\phi(t) &= 2\|\wedge^{-1}u_t\|^2 + 2\|u_t\|^2 + 2(\wedge^{-1}u_{tt}, \wedge^{-1}u) + 2(u_{tt}, u) \\
&= 2\|\wedge^{-1}u_t\|^2 + 2\|u_t\|^2 + 2(\wedge^{-2}u_{tt}, u) + 2(u_{tt}, u) \\
&= 2\|\wedge^{-1}u_t\|^2 + 2\|u_t\|^2 - 2\left(\|u\|_{H^1}^2 + \int_{-\infty}^{\infty} uf(u)\,\mathrm{d}x\right) \\
&= 2\|\wedge^{-1}u_t\|^2 + 2\|u_t\|^2 - 2I(u).
\end{aligned} \tag{4.5}$$

From (3.1) we get

$$\frac{1}{2}\|\wedge^{-1}u_t\|^2 + \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|u\|_{H^1}^2 + \frac{1}{p+1}I(u) = E(0).$$

Hence we have

$$\ddot\phi(t) = (p+3)\big(\|\wedge^{-1}u_t\|^2 + \|u_t\|^2\big) + (p-1)\|u\|_{H^1}^2 - 2(p+1)E(0). \tag{4.6}$$

(i) If $0 < E(0) < d$, then Theorem 3.2 gives $u(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$, where (δ_1, δ_2) is the maximal interval such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$. In particular, we have $u \in V$ for $0 \leqslant t < \infty$. Hence from Lemma 2.11 we get

$$(p-1)\|u\|_{H^1}^2 > 2(p+1)d > 2(p+1)E(0)$$

and

$$\ddot{\phi}(t) > (p+3)\left(\left\|\wedge^{-1}u_t\right\|^2 + \|u_t\|^2\right), \tag{4.7}$$

which gives

$$\phi(t)\ddot{\phi}(t) - \frac{p+3}{4}\left(\dot{\phi}(t)\right)^2 \geqslant (p+3)\left(\left(\left\|\wedge^{-1}u\right\|^2 + \|u\|^2\right)\left(\left\|\wedge^{-1}u_t\right\|^2 + \|u_t\|^2\right) - \left(\left(\wedge^{-1}u, \wedge^{-1}u_t\right) + (u, u_t)\right)^2\right).$$

From Schwatz inequality we get

$$\begin{aligned}\left(\left(\wedge^{-1}u, \wedge^{-1}u_t\right) + (u, u_t)\right)^2 &= \left(\wedge^{-1}u, \wedge^{-1}u_t\right)^2 + (u, u_t)^2 + 2\left(\wedge^{-1}u, \wedge^{-1}u_t\right)(u, u_t)\\ &\leqslant \left\|\wedge^{-1}u\right\|^2\left\|\wedge^{-1}u_t\right\|^2 + \|u\|^2\|u_t\|^2 + \left\|\wedge^{-1}u\right\|^2\|u_t\|^2 + \|u\|^2\left\|\wedge^{-1}u_t\right\|^2.\end{aligned}$$

Hence we have

$$\phi(t)\ddot{\phi}(t) - \frac{p+3}{4}\left(\dot{\phi}(t)\right)^2 \geqslant 0$$

and

$$\begin{aligned}&\left(\phi^{-\alpha}(t)\right)'' = \frac{-\alpha}{\phi(t)^{\alpha+2}}\left(\phi(t)\ddot{\phi}(t) - (\alpha+1)\left(\dot{\phi}(t)\right)^2\right) \leqslant 0,\\ &\alpha = \frac{p-1}{4}, \quad 0 \leqslant t < \infty.\end{aligned} \tag{4.8}$$

On the other hand, from $u \in V_\delta$ for $\delta \in (1, \delta_2)$ we get $I_\delta(u) < 0$ and $\|u\|_{H^1} > r(\delta)$ for $\delta \in (1, \delta_2)$, $0 \leqslant t < \infty$. Hence we have $I_{\delta_2}(u) \leqslant 0$ and $\|u\|_{H^1} \geqslant r(\delta_2)$ for $0 \leqslant t < \infty$. Thus from (4.5) we get

$$\begin{aligned}\ddot{\phi}(t) &\geqslant -2I(u) = 2(\delta_2 - 1)\|u\|_{H^1}^2 - 2I_{\delta_2}(u) \geqslant 2(\delta_2 - 1)\|u\|_{H^1}^2\\ &\geqslant 2(\delta_2 - 1)r^2(\delta_2), \quad 0 \leqslant t < \infty,\\ \dot{\phi}(t) &\geqslant 2(\delta_2 - 1)r^2(\delta_2)t + \dot{\phi}(0), \quad 0 \leqslant t < \infty.\end{aligned}$$

Hence there exists $t_0 \geqslant 0$ such that $\dot{\phi}(t) > \dot{\phi}(t_0) > 0$ for $t > t_0$ and

$$\phi(t) > \dot{\phi}(t_0)(t - t_0) + \phi(t_0) \geqslant \dot{\phi}(t_0)(t - t_0), \quad t > t_0.$$

Therefore there exits $t_1 \geqslant t_0$ such that $\phi(t_1) > 0$ and $\dot{\phi}(t_1) > 0$. Thus from (4.8) it follows that there exists $T_1 > 0$ such that

$$\lim_{t \to T_1} \phi^{-\alpha}(t) = 0$$

and

$$\lim_{t \to T_1} \phi(t) = +\infty, \tag{4.9}$$

which contradicts $T_m = +\infty$.

(ii) If $E(0) \leqslant 0$, then (4.6) gives (4.7) immediately. On the other hand, since $I(u_0) < 0$ implies $\|u_0\|_{H^1} \neq 0$, Theorem 3.6 gives $u \in V_\delta$ for $\delta \in (0, \frac{p+1}{2})$. Hence if in the proof of (i) δ_2 is replaced by $\frac{p+1}{2}$, then we can also prove that there exists $t_1 > 0$ such that $\phi(t_1) > 0$, $\dot{\phi}(t_1) > 0$ and (4.9), which contradicts $T_m = +\infty$. □

From Theorem 4.5 and Proposition 1.2 we can obtain the following corollary.

Corollary 4.6. *Under the conditions of Theorem* 4.5 *for the local weak solution u of problem* (1.4), (1.5) *we have*

$$\limsup_{t \to T_m} \|u\|_\infty = +\infty$$

and

$$\limsup_{t \to T_m}\left(\|u\|_{H^1}^2 + \|u_t\|^2\right) = +\infty, \tag{4.10}$$

where T_m is the maximal existence time of u.

From Theorems 4.1, 4.5 and Corollary 4.6 we can obtain the following threshold result of global existence and nonexistence of weak solution for problem (1.4), (1.5).

Theorem 4.7. *Let $f(u)=|u|^p$, $p\geqslant 2$, $u_0\in H^s$, $u_1\in H^{s-1}$ for some $s\geqslant 1$ and $\wedge^{-1}u_1\in L^2$. Assume that $E(0)<d$. Then when $I(u_0)\geqslant 0$, problem* (1.4), (1.5) *admits a unique global solution $u\in C([0,\infty);H^{s_1})\cap C^1([0,\infty),H^{s_1-1})\cap C^2([0,\infty),H^{s_1-2})$, where $s_1=\min\{s,p-1\}$; when $I(u_0)<0$, the problem does not admit any above global solution and the local solution blows up in finite time.*

From Theorems 4.2, 4.5 and Corollary 4.6 we can obtain the following threshold result of global existence and nonexistence of solution for problem (1.4), (1.5).

Theorem 4.8. *Let $f(u)=u^{2k}$, $k=1,2,\dots$, $u_0\in H^s$, $u_1\in H^{s-1}$ for some $s\geqslant 1$ and $\wedge^{-1}u_1\in L^2$. Assume that $E(0)<d$. Then when $I(u_0)\geqslant 0$, problem* (1.4), (1.5) *admits a unique global solution $u\in C([0,\infty);H^s)\cap C^1([0,\infty),H^{s-1})\cap C^2([0,\infty),H^{s-2})$; when $I(u_0)<0$, the problem does not admit any above global solution and the local solution blows up in finite time.*

5. Problem (1.4), (1.5) with critical initial condition $E(0)=d$

In this section we discuss the global existence of solution for problem (1.4), (1.5) with critical initial condition $E(0)=d$ and give a threshold result of global existence and nonexistence of solution. First we consider the case $E(0)=d$, $I(u_0)\geqslant 0$.

Theorem 5.1. *Let $f(u)=|u|^p$, $p\geqslant 2$, $u_0\in H^s$, $u_1\in H^{s-1}$ for some $s\geqslant 1$ and $\wedge^{-1}u_1\in L^2$. Assume that $E(0)=d$, $I(u_0)\geqslant 0$. Then problem* (1.4), (1.5) *admits a unique global solution $u\in C([0,\infty);H^{s_1})\cap C^1([0,\infty),H^{s_1-1})\cap C^2([0,\infty),H^{s_1-2})$ and $u\in\bar{W}=W\cap\partial W$ for $0\leqslant t<\infty$, where $s_1=\min\{s,p-1\}$.*

Proof. First from Corollary 1.3 it follows that problem (1.4), (1.5) admits a unique local solution $u\in C([0,T_m);H^{s_1})\cap C^1([0,T_m),H^{s_1-1})\cap C^2([0,T_m),H^{s_1-2})$, where T_m is the maximal existence time of u. Next we prove $T_m=+\infty$. In order to do this we first prove that problem (1.4), (1.5) admits a global weak solution. We prove this for two cases $\|u_0\|_{H^1}\neq 0$ and $\|u_0\|_{H^1}=0$.

(1) $\|u_0\|_{H^1}\neq 0$.

(i) If $I(u_0)>0$, then $I(\lambda u_0)|_{\lambda=1}=\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u_0)|_{\lambda=1}>0$. Hence there exists an interval (λ',λ'') such that $\lambda'<1<\lambda''$, $I(\lambda u_0)>0$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u_0)>0$ for $\lambda\in(\lambda',\lambda'')$. Take a sequence $\{\lambda_m\}$ such that $\lambda'<\lambda_m<1$, $m=1,2,\dots,$ and $\lambda_m\to 1$ as $m\to\infty$. Let $u_{0m}(x)=\lambda_m u_0(x)$, $m=1,2,\dots$. Consider the initial conditions

$$u(x,0)=u_{0m}(x),\quad u_t(x,0)=u_1(x) \tag{5.1}$$

and corresponding problem (1.4), (5.1). Then we have

$$I(u_{0m})=I(\lambda_m u_0)>0 \tag{5.2}$$

and

$$\begin{aligned}E_m(0)&=\frac12\left\|\wedge^{-1}u_1\right\|^2+\frac12\|u_1\|^2+J(u_{0m})=\frac12\left\|\wedge^{-1}u_1\right\|^2+\frac12\|u_1\|^2+J(\lambda_m u_0)\\&<\frac12\left\|\wedge^{-1}u_1\right\|^2+\frac12\|u_1\|^2+J(u_0)=E(0)=d.\end{aligned} \tag{5.3}$$

(ii) If $I(u_0)=0$, then $u_0\in\mathcal{N}$ and by the definition of d we have $J(u_0)\geqslant d$, which together with

$$\frac12\left\|\wedge^{-1}u_1\right\|^2+\frac12\|u_1\|^2+J(u_0)=E(0)=d$$

gives $J(u_0)=d$. From Lemma 2.12 we have $u_0\in\mathcal{N}^-$. Hence form Corollary 2.5 it follows that $\lambda^*=\lambda^*(u_0)=1$, $J(\lambda u_0)$ is increasing and $I(\lambda u_0)>0$ for $0<\lambda<1$. Take a sequence $\{\lambda_m\}$ such that $0<\lambda_m<1$, $m=1,2,\dots,$ and $\lambda_m\to 1$ as $m\to\infty$. Again let $u_{0m}(x)=\lambda_m u_0(x)$ and consider problem (1.4), (5.1). Then (5.2) and (5.3) also hold.

From Theorem 4.1 it follows that for each m problem (1.4), (5.1) admits a unique global solution $u_m \in C([0,\infty); H^{s_1}) \cap C^1([0,\infty), H^{s_1-1}) \cap C^2([0,\infty), H^{s_1-2})$ with $\wedge^{-1}u_{mt} \in C([0,\infty); L^2)$ and $u_m \in W$ for $0 \leqslant t < \infty$ satisfying (by (4.4))

$$\big(\wedge^{-1}u_{mt}, \wedge^{-1}v\big) + (u_{mt}, v) + \int_0^t \big((u_m, v)_{H^1} + \big(f(u_m), v\big)\big)\,\mathrm{d}\tau = \big(\wedge^{-1}u_1, \wedge^{-1}v\big) + (u_1, v),$$

$$\forall v \in H^1,\ 0 \leqslant t < \infty, \tag{5.4}$$

and

$$\frac{1}{2}\big\|\wedge^{-1}u_{mt}\big\|^2 + \frac{1}{2}\|u_{mt}\|^2 + J(u_m) = E_m(0) < d. \tag{5.5}$$

From (5.5) and

$$J(u_m) = \frac{p-1}{2(p+1)}\|u_m\|^2_{H^1} + \frac{1}{p+1}I(u_m) \geqslant \frac{p-1}{2(p+1)}\|u_m\|^2_{H^1},$$

we get

$$\frac{1}{2}\big\|\wedge^{-1}u_{mt}\big\|^2 + \frac{1}{2}\|u_{mt}\|^2 + \frac{p-1}{2(p+1)}\|u_m\|^2_{H^1} < d, \quad 0 \leqslant t < \infty. \tag{5.6}$$

From (5.6) we obtain

$$\|u_m\|^2_{H^1} \leqslant \frac{2(p+1)}{p-1}d, \quad 0 \leqslant t < \infty, \tag{5.7}$$

$$\big\|\wedge^{-1}u_{mt}\big\|^2 + \|u_{mt}\|^2 \leqslant 2d, \quad 0 \leqslant t < \infty, \tag{5.8}$$

$$\|u_m\|^2_{p+1} \leqslant C_*^2\|u_m\|^2_{H^1} \leqslant C_*^2\frac{2(p+1)}{p-1}d, \quad 0 \leqslant t < \infty, \tag{5.9}$$

$$\big\|f(u_m)\big\|_r^r = \|u_m\|^{p+1}_{p+1} \leqslant C_*^{p+1}\|u_m\|^{p+1}_{H^1} \leqslant C_*^{p+1}\left(\frac{2(p+1)}{p-1}d\right)^{\frac{p+1}{2}}, \quad r = \frac{p+1}{p},\ 0 \leqslant t < \infty. \tag{5.10}$$

From (5.7)–(5.10) it follows that there exist $\bar{u} \in \bar{W}$ and a subsequence $\{u_\nu\}$ of $\{u_m\}$ such that as $\nu \to \infty$,

$u_\nu \to \bar{u}$ in $L^\infty(0,\infty; H^1)$ weakly star and a.e. in $\mathbb{R} \times [0,\infty)$,
$u_{\nu t} \to \bar{u}_t$ in $L^\infty(0,\infty; L^2)$ weakly star,
$\wedge^{-1}u_{\nu t} \to \wedge^{-1}\bar{u}_t$ in $L^\infty(0,\infty; L^2)$ weakly star,
$f(u_\nu) \to f(\bar{u})$ in $L^\infty(0,\infty; L^r)$ weakly star.

In (5.4) letting $m = \nu \to \infty$, we get

$$\big(\wedge^{-1}\bar{u}_t, \wedge^{-1}v\big) + (\bar{u}_t, v) + \int_0^t \big((\bar{u}, v)_{H^1} + \big(f(\bar{u}), v\big)\big)\,\mathrm{d}\tau = \big(\wedge^{-1}u_1, \wedge^{-1}v\big) + (u_1, v), \tag{5.11}$$

which implies $\bar{u}$ satisfies (1.4). In addition, clearly we have $\bar{u}(x,0) = u_0(x)$ and $\bar{u}_t(x,0) = u_1(x)$. Thus $\bar{u}$ is a global weak solution of problem (1.4), (1.5). From the uniqueness of weak solution for problem (1.4), (1.5), we get $\bar{u} = u$ on $\mathbb{R} \times [0, T_m)$. Hence from

$$\|u\|^2_{H^1} = \|\bar{u}\|^2_{H^1} \leqslant C, \quad 0 \leqslant t < T_m,$$

we obtain $T_m = +\infty$ and $u \in C([0,\infty); H^{s_1}) \cap C^1([0,\infty), H^{s_1-1}) \cap C^2([0,\infty), H^{s_1-2})$.

(2) $\|u_0\|_{H^1} = 0$.

In this case we have $J(u_0) = 0$. Hence from

$$\frac{1}{2}\left\|\wedge^{-1}u_1\right\|^2 + \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) = d,$$

we get $\frac{1}{2}\| \wedge^{-1} u_1\|^2 + \frac{1}{2}\|u_1\|^2 = d$. Take a sequence $\{\lambda_m\}$ such that $0 < \lambda_m < 1$, $m = 1, 2, \ldots,$ and $\lambda_m \to 1$ as $m \to \infty$. Let $u_{1m}(x) = \lambda_m u_1(x)$. Consider the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_{1m}(x) \tag{5.12}$$

and corresponding problem (1.4), (5.12). Then

$$\begin{aligned}
E_m(0) &= \frac{1}{2}\left\|\wedge^{-1}u_{1m}\right\|^2 + \frac{1}{2}\|u_{1m}\|^2 + J(u_0) \\
&= \frac{1}{2}\left\|\wedge^{-1}(\lambda_m u_1)\right\|^2 + \frac{1}{2}\|\lambda_m u_1\|^2 \\
&< \frac{1}{2}\left\|\wedge^{-1}u_1\right\|^2 + \frac{1}{2}\|u_1\|^2 = d.
\end{aligned}$$

Hence from Theorem 4.1 it follows that for each m problem (1.4), (5.12) admits a unique global solution $u_m \in C([0,\infty); H^{s_1}) \cap C^1([0,\infty), H^{s_1-1}) \cap C^2([0,\infty), H^{s_1-2})$ with $\wedge^{-1}u_{mt} \in C([0,\infty); L^2)$ and $u_m \in W$ for $0 \leqslant t < \infty$ satisfying (5.4) and (5.5). The remainder of the proof is the same as that in the proof of part (i). $\square$

By a similar argument from Theorem 5.1 and Corollary 1.4 we can obtain the following theorem.

Theorem 5.2. *Let $f(u) = u^{2k}$, $k = 1, 2, \ldots,$ $u_0 \in H^s$, $u_1 \in H^{s-1}$ for some $s \geqslant 1$ and $\wedge^{-1}u_1 \in L^2$. Assume that $E(0) = d$, $I(u_0) \geqslant 0$. Then problem* (1.4), (1.5) *admits a unique global solution $u \in C([0,\infty); H^s) \cap C^1([0,\infty); H^{s-1}) \cap C^2([0,\infty); H^{s-2})$ and $u \in \bar{W}$ for $0 \leqslant t < \infty$.*

Next we discuss the global existence of solution for problem (1.4), (1.5) with $E(0) = d$, $I(u_0) < 0$.

Lemma 5.3. *Let $f(u) = |u|^p$, $p \geqslant 2$, $u_0 \in H^1$, $u_1 \in L^2$, $\wedge^{-1}u_1 \in L^2$ and*

$$V' = \left\{u \in H^1 \mid I(u) < 0\right\}.$$

Assume that $E(0) = d$ and $(\wedge^{-1}u_0, \wedge^{-1}u_1) + (u_0, u_1) \geqslant 0$. Then V' is invariant under the flow of (1.4), (1.5).

Proof. Let $u \in C([0,T_m); H^1) \cap C^1([0,T_m); L^2) \cap C^2([0,T_m); H^{-1})$ with $\wedge^{-1}u_t \in C([0,T_m); L^2)$ be any weak solution of problem (1.4), (1.5) with $I(u_0) < 0$, $E(0) = d$ and $(\wedge^{-1}u_0, \wedge^{-1}u_1) + (u_0, u_1) \geqslant 0$, T_m be the maximal existence time of u. We prove $I(u) < 0$ for $0 < t < T_m$. If it is false, then there exits $t_0 \in (0, T_m)$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $0 \leqslant t < t_0$. From Lemma 2.7 we have $\|u\|_{H^1} > r(1)$ for $0 \leqslant t < t_0$ and $\|u(t_0)\|_{H^1} \geqslant r(1)$. Hence we get $J(u(t_0)) \geqslant d$ which together with

$$\frac{1}{2}\left\|\wedge^{-1}u_t(t_0)\right\|^2 + \frac{1}{2}\left\|u_t(t_0)\right\|^2 + J\big(u(t_0)\big) = E(0) = d$$

give $J(u(t_0)) = d$, $\frac{1}{2}\| \wedge^{-1} u_t(t_0)\|^2 + \frac{1}{2}\|u_t(t_0)\|^2 = 0$ and $(\wedge^{-1}u(t_0), \wedge^{-1}u_t(t_0)) = (u(t_0), u_t(t_0)) = 0$. On the other hand, let

$$\phi(t) = \left\|\wedge^{-1}u\right\|^2 + \|u\|^2.$$

Then

$$\begin{aligned}
&\dot{\phi}(t) = 2\big(\wedge^{-1}u, \wedge^{-1}u_t\big) + 2(u, u_t) \quad \text{with } \dot{\phi}(0) = 2\big(\wedge^{-1}u_0, \wedge^{-1}u_1\big) + 2(u_0, u_1) \geqslant 0, \\
&\ddot{\phi}(t) = 2\left\|\wedge^{-1}u_t\right\|^2 + 2\|u_t\|^2 - 2I(u) > 0, \quad 0 \leqslant t < t_0.
\end{aligned}$$

Hence $\dot{\phi}(t)$ is strictly increasing on $0 \leqslant t \leqslant t_0$ and $\dot{\phi}(t_0) = 2(\wedge^{-1}u(t_0), \wedge^{-1}u_t(t_0)) + 2(u(t_0), u_t(t_0)) > 0$, which contradicts above conclusion. $\square$

Theorem 5.4. *Let $f(u)=|u|^p$, $p\geqslant 2$, $u_0\in H^1$, $u_1\in L^2$ and $\wedge^{-1}u_1\in L^2$. Assume that $E(0)=d$, $I(u_0)<0$ and $(\wedge^{-1}u_0,\wedge^{-1}u_1)+(u_0,u_1)\geqslant 0$. Then the existence time of weak solution for problem* (1.4), (1.5) *is finite.*

Proof. First form Corollary 1.3 it follows that problem (1.4), (1.5) admits a unique local weak solution $u\in C([0,T_m);H^1)\cap C^1([0,T_m);L^2)\cap C^2([0,T_m);H^{-1})$, where T_m is the maximal existence time of u. We prove $T_m<\infty$. If it is false, then $T_m=+\infty$. Again let

$$\phi(t)=\|\wedge^{-1}u\|^2+\|u\|^2.$$

Then we have

$$\dot\phi(t)=2(\wedge^{-1}u_t,\wedge^{-1}u)+2(u,u_t).$$

And by Lemma 5.3 we get

$$\ddot\phi(t)=2\|\wedge^{-1}u_t\|^2+2\|u_t\|^2-2I(u)>0,\quad 0\leqslant t<\infty.$$

Finally we get

$$\begin{aligned}\ddot\phi(t)&=(p+3)\left(\|\wedge^{-1}u_t\|^2+\|u_t\|^2\right)+(p-1)\|u\|_{H^1}^2-2(p+1)E(0)\\&=(p+3)\left(\|\wedge^{-1}u_t\|^2+\|u_t\|^2\right)+(p-1)\|u\|_{H^1}^2-2(p+1)d,\quad 0\leqslant t<\infty.\end{aligned}$$

From Lemma 5.3 we have $I(u)<0$ for $0\leqslant t<\infty$ hence by Lemma 2.11 we get

$$(p-1)\|u\|_{H^1}^2>2(p+1)d.$$

Hence we have

$$\ddot\phi(t)>(p+3)\left(\|\wedge^{-1}u_t\|^2+\|u_t\|^2\right),\quad 0\leqslant t<\infty,$$

which gives (4.8). On the other hand, from $\ddot\phi(t)>0$ and $\dot\phi(0)\geqslant 0$ we can get $\dot\phi(t)>0$ and $\phi(t)>0$ for all $t>0$. Therefore there exists $T_1>0$ such that

$$\lim_{t\to T_1}\phi(t)=+\infty,$$

which contradicts $T_m=+\infty$. □

From Theorem 5.4 and Corollary 1.3 we can obtain the following corollary.

Corollary 5.5. *Under the conditions of Theorem* 5.4, *the weak solution u of problem* (1.4), (1.5) *blows up in finite time, i.e., if T_m is the maximal existence time of u, then*

$$\limsup_{t\to T_m}\|u\|_\infty=+\infty$$

and

$$\limsup_{t\to T_m}\|u\|_{H^1}^2=+\infty.$$

From Theorems 5.1, 5.2, 5.4 and Corollary 5.5 we can obtain the following threshold results of global existence and nonexistence of solution for problem (1.4), (1.5) with $E(0)=d$.

Corollary 5.6. *Let $f(u)=|u|^p$, $p\geqslant 2$, $u_0\in H^s$, $u_1\in H^{s-1}$ for some $s\geqslant 1$ and $\wedge^{-1}u_1\in L^2$. Assume that $E(0)=d$ and $(\wedge^{-1}u_0,\wedge^{-1}u_1)+(u_0,u_1)\geqslant 0$. Then when $I(u_0)\geqslant 0$, problem* (1.4), (1.5) *admits a unique global solution $u\in C([0,\infty);H^{s_1})\cap C^1([0,\infty),H^{s_1-1})\cap C^2([0,\infty),H^{s_1-2})$, where $s_1=\min\{s,p-1\}$; when $I(u_0)<0$ the problem does not admit any above global solution and the local solution blows up in finite time.*

Corollary 5.7. *Let $f(u) = u^{2k}$, $k = 1, 2, \ldots$, $u_0 \in H^s$, $u_1 \in H^{s-1}$ for some $s \geqslant 1$ and $\wedge^{-1}u_1 \in L^2$. Assume that $E(0) = d$ and $(\wedge^{-1}u_0, \wedge^{-1}u_1) + (u_0, u_1) \geqslant 0$. Then when $I(u_0) \geqslant 0$, problem* (1.4), (1.5) *admits a unique global solution $u \in C([0,\infty); H^s) \cap C^1([0,\infty), H^{s-1}) \cap C^2([0,\infty), H^{s-2})$; when $I(u_0) < 0$ the problem does not admit any above global solution and the local solution blows up in finite time.*

Remark 5.8. Note that all of the proof of the main results in this paper strongly depends on the introducing of families $\{W_\delta\}$ and $\{V_\delta\}$. Therefore in order to obtain the results in this paper, the introducing of families $\{W_\delta\}$ and $\{V_\delta\}$ is crucial.

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